Inter (Part-II) 2021

Mathematics	(Group-I)	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: (16)

(i) Find the domain and range of the function g defined by: $g(x) = \sqrt{x^2 - 4}$

Ans Domain g = $(-\infty, -2) \cup (2, \infty)$

$$(\because \; x^2 - 4 \geq 0 \Rightarrow x^2 \geq 4 \Rightarrow x \leq -2 \text{ or } x \geq 2)$$

Range g = $(0, \infty)$

$$(\because \; x^2 - 4 = 0 \text{ if } x = \pm 2)$$

(ii) The real valued functions f and g are given. Find fog (x), if

$$f(x) = 3x^4 - 2x^2 \quad \text{and} \quad g(x) = \frac{2}{\sqrt{x}}, \; x \neq 0$$

Ans $fog(x) = f(g(x))$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2 \quad \text{(Using the rule of f)}$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x} \Rightarrow \frac{48 - 8x}{x^2}$$

$$= \frac{8(6 - x)}{x^2}, \quad x \neq 0$$

(iii) Evaluate $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$.

Ans $$\frac{1 - \cos\theta}{\theta} = \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)}$$

$$= \frac{\sin^2\theta}{\theta(1 + \cos\theta)}$$

$$= \sin\theta\left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{1 + \cos\theta}\right)$$

$$\therefore \quad \lim_{\theta\to 0}\frac{1 - \cos\theta}{\theta} = \lim_{\theta\to 0}\sin\theta \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\left(\frac{1}{1 + \cos\theta}\right)$$

$$= 0(1)\left(\frac{1}{1 + 1}\right)$$

$$= 0$$

(iv) **Evaluate $\lim_{x\to 1}\frac{x^3 - 3x^2 + 3x - 1}{x^3 - x}$.**

Ans If we put $x = 1$, the above expression will be $\frac{0}{0}$, so firstly we factorize numerator and denominator.

$$x^3 - 3x^2 + 3x - 1 = (x)^3 - 3x(x - 1) - (1)^3$$
$$= (x - 1)^3$$
$$= (x - 1)(x - 1)^2$$
$$= (x - 1)(x^2 - 2x + 1)$$

and
$$x^3 - x = x(x^2 - 1)$$
$$= x(x + 1)(x - 1)$$

$$\text{So, } \lim_{x\to 1}\frac{x^3 - 3x^2 + 3x - 1}{x^3 - x} = \lim_{x\to 1}\frac{(x - 1)(x^2 - 2x + 1)}{x(x + 1)(x - 1)}$$

$$= \lim_{x\to 1}\frac{x^2 - 2x + 1}{x(x + 1)}$$

$$= \frac{\lim_{x\to 1}(x^2 - 2x + 1)}{\lim_{x\to 1}x(x + 1)}$$

$$= \frac{(1)^2 - 2(1) + 1}{1(1 + 1)}$$

$$= \frac{1 - 2 + 1}{1(2)} = \frac{0}{2}$$

$$= 0$$

(v) **Find $\frac{dy}{dx}$ if $x^2 - 4xy - 5y = 0$.**

Ans $x^2 - 4xy - 5y = 0$

$$\frac{d}{dx}(x^2 - 4xy - 5y) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(4xy) - \frac{d}{dx}(5y) = 0$$

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) - 5\frac{d}{dx}(y) = 0$$

$$2x - 4\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] - 5\frac{dy}{dx} = 0$$

$$2x - 4x\frac{dy}{dx} - 4y - 5\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx} + 5\frac{dy}{dx} = 2x - 4y$$

$$\frac{dy}{dx}(4x + 5) = 2(x - 2y)$$

$$\frac{dy}{dx} = \frac{2(x - 2y)}{4x + 5}$$

(vi) **Differentiate w.r.t. 'x' $\cot^{-1}\left(\frac{x}{a}\right)$.**

Ans Let $y = \cot^{-1}\left(\frac{x}{a}\right)$

So,
$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}\left(\frac{x}{a}\right)\right]$$

$$= -\frac{1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= -\frac{1}{1 + \frac{x^2}{a^2}}\left(\frac{1}{a}\right)\frac{d}{dx}(x)$$

$$= -\frac{1}{\frac{a^2 + x^2}{a^2}}\left(\frac{1}{a}\right)(1)$$

$$= -\frac{a^2}{a^2 + x^2}\left(\frac{1}{a}\right)$$

$$= \frac{a}{a^2 + x^2}$$

(vii) **Find $f'(x)$ if $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$.**

Ans Let $u = e^{2x} + e^{-2x}$

Then $\quad f(x)$ becomes

$$f(x) = \sqrt{ln\, u} = (ln\, u)^{1/2}$$

As $\quad f'(x) = \frac{dy}{dx}$

$$= \frac{dy}{du} \times \frac{du}{dx} \qquad \text{(By chain rule)}$$

$$= \frac{d}{du}(ln\, u)^{1/2} \times \frac{d}{dx} u$$

$$= \left[\frac{1}{2}(ln\, u)^{1/2-1} \cdot \frac{d}{du}(ln\, u)\right] \times \frac{d}{dx}(e^{2x} + e^{-2x})$$

$$= \left(\frac{1}{2}(ln\, u)^{-1/2} \cdot \frac{1}{u}\right) \cdot \left(e^{2x} \cdot 2 + e^{-2x}(-2)\right)$$

$$= \left(\frac{1}{2(ln\, u)^{1/2}} \cdot \frac{1}{u}\right) \cdot (2)(e^{2x} - e^{-2x})$$

$$= \frac{2}{2} \cdot \left(\frac{1}{\sqrt{ln\, u}} \cdot \frac{1}{u}\right)(e^{2x} - e^{-2x})$$

$$= \frac{1}{\sqrt{ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{(e^{2x} + e^{-2x})}(e^{2x} - e^{-2x})$$

$$= \frac{e^{2x} - 2^{-2x}}{\sqrt{ln(e^{2x} + e^{-2x})}\,(e^{2x} + e^{-2x})}$$

(viii) Find y_2 if $x^3 - y^3 = a^3$.

Ans Given: $\quad x^3 - y^3 = a^3 \qquad$ (i)

Differentiating (i) w.r.t 'x', we get

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) = \frac{d}{dx}(a^3)$$

$$3x^2 - 3y^2 \frac{dy}{dx} = 0$$

$$-3y^2 \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{-3y^2}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

Differentiating again of above expression:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$= \frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{(y^2)^2}$$

$$= \frac{y^2(2x) - x^2(2y)\frac{dy}{dx}}{y^4}$$

$$= \frac{2xy\left[y - x\left(\frac{dy}{dx}\right)\right]}{y^4}$$

$$= \frac{2xy\left[y - x\left(\frac{x^2}{y^2}\right)\right]}{y^4} \qquad \because \left(\frac{dy}{dx} = \frac{x^2}{y^2}\right)$$

$$= \left(\frac{y^2}{y^2}\right)\frac{2xy\left[y - x\left(\frac{x^2}{y^2}\right)\right]}{y^4}$$

$$= \frac{2xy\left[y \,.\, y^2 - x.y^2\left(\frac{x^2}{y^2}\right)\right]}{y^2y^4}$$

$$= \frac{2x\,(y^3 - x^3)}{y \,.\, y^4}$$

$$= \frac{2x\,(-1)\,(-y^3 + x^3)}{y^5}$$

$$= \frac{-2x\,(x^3 - y^3)}{y^5}$$

$$= \frac{-2x\,(a^3)}{y^5} \qquad [\because \text{ By the given expression}]$$

$$= \frac{-2a^3x}{y^5}$$

(ix) **Prove that $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$**

Ans Let $\quad y = \text{cosec}^{-1} x \qquad$ (i)

Then,

$x = \text{cosec } y \quad$ or $\quad x = \text{cosec } y$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \qquad$ (ii)

$-\left(\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ is also written as $\left[-\frac{\pi}{2}, 0\right[\cup \left]0, \frac{\pi}{2}\right]$

Differentiating both sides of (ii) w.r.t. 'x', we get

$$1 = \frac{d}{dx}(\text{cosec } y) = \frac{d}{dy}(\text{cosec } y)\frac{dy}{dx}$$

$$= (-\text{cosec } y \cot y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\text{cosec } y \cot y} \quad \text{for} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$$

When $y \in \left(0, \frac{\pi}{2}\right)$, cosec y and cot y are positive.

As cosec y = x, so x is positive in this case

and $\cot y = \sqrt{\text{cosec}^2 y - 1} = \sqrt{x^2 - 1}$, for all $x > 1$

Thus $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$, for $x > 1$

(x) Differentiate $\frac{2x-1}{\sqrt{x^2+1}}$.

Ans Let $y = \frac{2x-1}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-1}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}\frac{d}{dx}(2x-1) - (2x-1)\frac{d}{dx}(x^2+1)^{1/2}}{(\sqrt{x^2+1})^2}$$

$$= \frac{\sqrt{x^2+1}\,(2) - (2x-1)\frac{1}{2}(x^2+1)^{1/2-1}\frac{d}{dx}(x^2+1)}{x^2+1}$$

$$= \frac{2\sqrt{x^2+1} - (2x-1)\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$

$$= \left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)\frac{2\sqrt{x^2+1} - (2x-1)\frac{1}{\sqrt{x^2+1}}(x)}{x^2+1}$$

$$= \frac{2\sqrt{x^2+1}\sqrt{x^2+1} - x(2x-1)\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}}{\sqrt{x^2+1}\,(x^2+1)}$$

$$= \frac{2\left(\sqrt{x^2+1}\right)^2 - x(2x-1)}{(x^2+1)^{1/2}(x^2+1)}$$

$$= \frac{2(x^2+1) - x(2x-1)}{(x^2+1)^{1/2+1}}$$

$$= \frac{2x^2+2-2x^2+x}{(x^2+1)^{3/2}}$$

$$= \frac{x+2}{(x^2+1)^{3/2}}$$

(xi) Find the interval in which function is increasing or decreasing:

$$\mathbf{f(x) = \cos x \quad x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$$

Ans $f(x) = \cos x \quad x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$f'(x) = -\sin x$

$f'(x) = -(\sin x)$ is positive for $\frac{-\pi}{2} < x < 0$ as $\sin x$ is negative for $\frac{-\pi}{2} < x < 0$, so f is increasing on the interval $\left(\frac{-\pi}{2}, 0\right)$.

Similarly, $f'(x) = -\sin x$ is negative for $0 < x < \frac{\pi}{2}$ as $\sin x$ is positive for $0 < x < \frac{\pi}{2}$, so f is decreasing on the interval $\left(0, \frac{\pi}{2}\right)$.

(xii) Find y_4 if $y = \sin 3x$

Ans Given $y = \sin 3x$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$= \cos 3x \frac{d}{dx}(3x)$$

$$= \cos 3x\,(3)$$

$$= 3\cos 3x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}(y_1)$$

$$= \frac{d}{dx}(3\cos 3x) = 3\frac{d}{dx}(\cos 3x)$$

$$= 3(-\sin 3x)\frac{d}{dx}(3x)$$

$$= -3\sin 3x\,(3)$$

$$= -9\sin 3x$$

$$y_3 = \frac{d^3y}{dx^3} = \frac{d}{dx}(y_2)$$

$$= \frac{d}{dx}(-9\sin 3x) = -9\frac{d}{dx}(\sin 3x)$$

$$= -9(\cos 3x)\frac{d}{dx}(3x)$$

$$= -9(\cos 3x)(3)$$

$$= -27\cos 3x$$

$$y_4 = \frac{d^4y}{dx^4} = \frac{d}{dx}(y_3)$$

$$= \frac{d}{dx}(-27\cos 3x) = -27\frac{d}{dx}(\cos 3x)$$

$$= -27(-\sin 3x)\frac{d}{dx}(3x)$$

$$= 27\sin 3x\,(3)$$

$$= 81\sin 3x$$

3. Write short answers to any EIGHT (8) questions: (16)

(i) Use differentials to approximate the value of $\sqrt[4]{17}$.

Ans Let $f(x) = x^{1/4}$

Then $f(x + \delta x) = (x + \delta x)^{1/4}$

$= (x + dx)^{1/4}$ $\quad (\because \ \delta x = dx)$

As the nearest perfect fourth root to 17 is 16, *i.e.*, $(2)^4$, so we take $x = 16$ and $dx = 1$.

Then $f(16) = (16)^{1/4}$

$= (2^4)^{1/4} = 2$

Now $f'(x) = \left(\frac{1}{4}\right) x^{-3/4}$

So, $f'(16) = \left(\frac{1}{4}\right)((16)^{1/4})^{-3}$

$= \left(\frac{1}{4}\right)((2^4)^{1/4})^{-3} = \frac{1}{4}(2)^{-3}$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{2^3}\right)$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{8}\right) = \frac{1}{32}$$

Using $f(x + \delta x) \approx f(x) + dy$, we get

$$f(x + \delta x) \approx f(x) + f'(x)\,dx$$

$$f(16 + 1) \approx f(16) + f'(16)\,(1)$$

$$\approx 2 + \frac{1}{32}(1) = 2 + 0.3125$$

$$= 2.3125$$

Thus, $\sqrt[4]{17} = 2.03125$

(ii) Solve $\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}$.

Ans $\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}$, $(x > 0)$

Rationalizing the denominator, we have

$$\int \frac{dx}{\sqrt{x+1} - \sqrt{x}} = \int \frac{\sqrt{x+1} + \sqrt{x}}{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}\,dx$$

$$= \int \frac{\sqrt{x+1} + \sqrt{x}}{x + 1 - x}\,dx = \int [(x+1)^{1/2} + x^{1/2}]\,dx$$

$$= \int (x+1)^{1/2}\,dx + \int x^{1/2}\,dx$$

$$= \frac{(x+1)^{3/2}}{\frac{3}{2}} + \frac{x^{3/2}}{\frac{3}{2}} + c = \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + c$$

(iii) Evaluate $\int \frac{\cot\sqrt{x}}{\sqrt{x}}\,dx$.

Ans Put $\sqrt{x} = z$,

Then $d(\sqrt{x}) = dz \Rightarrow \frac{1}{2\sqrt{x}}\,dx = dz$

or $\frac{1}{\sqrt{x}}\,dx = 2\,dz$

Thus $\int \frac{\cot\sqrt{x}}{\sqrt{x}}\,dx = \int \cot\sqrt{x} \cdot \frac{1}{\sqrt{x}}\,dx = \int \cot z \cdot (2\,dz)$

$$= 2\int \cot z dz = 2\int \frac{\cos z}{\sin z}\,dz = 2\int (\sin z)^{-1} \cos z\,dz$$

$$= 2\,ln\,|\sin z| + c, \quad (z > 0 \text{ as } x > 0)$$

$= 2\ ln\ |\sin\sqrt{x}| + c$

(iv) Solve $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$.

Ans $$I = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx \qquad \text{(i)}$$

Put $u = \sqrt{\tan x} = (\tan x)^{1/2}$

$$dx = \frac{1}{2}(\tan x)^{1/2-1} \frac{d}{dx}(\tan x)$$

$$= \frac{1}{2(\tan x)^{1/2}} \sec^2 x\, dx$$

$$dx = \frac{1}{2}\frac{\sec^2 x}{\sqrt{\tan x}} dx \qquad \text{(ii)}$$

From (i),

$$I = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx = \frac{2}{2}\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$$= 2\int \frac{1}{2}\frac{\sec^2 x}{\sqrt{\tan x}} dx$$

By putting dx from (ii), we get

$$I = 2\int dx$$

$$= 2u + c$$

$$= 2\sqrt{\tan x} + c$$

(v) Solve $\int e^{2x}[-\sin x + 2\cos x]\, dx$.

Ans Let $I = \int e^{2x}[-\sin x + 2\cos x]\, dx$

$$= \int e^{2x}(2\cos x - \sin x)\, dx$$

$$= \int e^{2x}[2\cos x + (-\sin x)]\, dx$$

Again let $f(x) = \cos x$

Then $f'(x) = -\sin x$

Thus, $$I = \int e^{2x}[2f(x) + f'(x)]\, dx \qquad \text{(iii)}$$

Since $\frac{d}{dx}(e^{2x} f(x)) = e^{2x} . 2f(x) + e^{2x} . f'(x)$

$$= e^{2x}[2f(x) + f'(x)]$$

So, by putting in (iii), we have

$$I = \int e^{2x} f(x)\, dx$$

$$= e^{2x} f(x) + c$$

$= e^{2x} \cos x + c.$

(vi) **Evaluate $\int_0^{\pi/4} \sec x (\sec x + \tan x)\, dx$.**

Ans $\int_0^{\pi/4} \sec x (\sec x + \tan x)\, dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx$

$= \int_0^{\pi/4} \sec^2 x\, dx + \int_0^{\pi/4} \sec x \tan x\, dx$

$= [\tan x]_0^{\pi/4} + [\sec x]_0^{\pi/4} = \left(\tan \frac{\pi}{4} - \tan 0\right) + \left(\sec \frac{x}{4} - \sec 0\right)$

$= (1 - 0) + (\sqrt{2} - 1) = \sqrt{2}$

(vii) **Solve the differential equation $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1 + y^2)$.**

Ans Given $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1 + y^2)$

Separating variables, we get

$2\frac{1}{1 + y^2}\, dy = x\, dx$

Integrating both sides, we get

$2\int \frac{1}{1 + y^2}\, dy = \int x\, dx$

$2 \tan^{-1} y = \frac{x^2}{2} + c_1$

$\tan^{-1} y = \frac{x^2}{2(2)} + \frac{c_1}{2}$

$\tan^{-1} y = \frac{x^2}{4} + c \qquad \because \left[\frac{c_1}{2} = c\right]$

(viii) **Evaluate $\int x\, ln x\, dx$.**

Ans Let $I = \int x \ln x\, dx$

$= \int (ln\, x) . dx$

$= \frac{x^2}{2} ln\, x - \int \frac{x^2}{2}\left(\frac{1}{x}\right) dx$

$= \frac{1}{2} x^2\, ln\, x - \frac{1}{2}\int x\, dx$

$= \frac{1}{2} x^2\, ln\, x - \frac{1}{2}\frac{x^2}{2} + c$

$$= \frac{1}{2}x^2\left[\ln x - \frac{1}{2}\right] + c$$

(ix) The points A(–5, –2), B(5, –4) are ends of a diameter of a circle. Find centre and radius of it.

Ans As the given points A(–5, –2) and B(5, –4) are ends of a diameter, so the centre C of the circle is their mid-points. Thus, C is:

$$\left(\frac{-5+5}{2}, \frac{-2+(-4)}{2}\right) = (0, -3)$$

And the radius of the circle is

$$\text{Radius} = (CA)$$
$$= \sqrt{(-5-0)^2 + (-2\,(-3))^2}$$
$$= \sqrt{(-5)^2 + (-2+3)^2}$$
$$= \sqrt{25+1}$$
$$= \sqrt{26}$$

(x) Transform the equation 5x – 12y + 39 = 0 into normal form.

Ans $5x - 12y = -39$. Divide both sides by $\pm\sqrt{5^2 + 12^3} = \pm 13$. Since R.H.S is to be positive, we have to take negative sign.

Hence $\frac{5x}{-13} + \frac{12y}{13} = 3$ is the normal form of the equation.

(xi) Find k so that the lines joining A(7, 3), B(k, –6) and C(–4, 5), D(–6, 4) are parallel.

Ans Slope of the line through A and B is:

$$= \frac{-6-3}{k-7} = -\frac{9}{k-7}$$

Slope of the line through C and D is:

$$= \frac{4-5}{-6(-4)} \Rightarrow \frac{4-5}{-6+4} = \frac{-1}{-2} \Rightarrow \frac{1}{2}$$

As the both lines are parallel, so their slopes are also equal

$$-\frac{9}{k-7} = \frac{1}{2}$$
$$-9(2) = k-7\ (1)$$
$$\Rightarrow \quad k - 7 = -18$$
$$k = -18 + 7$$
$$k = -11$$

(xii) Find the lines represented by $2x^2 + 3xy - 5y^2 = 0$

Ans The above equation can be written as:

$$5\left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2 = 0 \qquad \text{(i)}$$

Solving (i) for $\frac{y}{x}$, we get

$$\frac{y}{x} = \frac{3 \pm \sqrt{9+40}}{10} = \frac{3 \pm \sqrt{49}}{10}$$

$$\frac{y}{x} = \frac{3 \pm 7}{10}$$

$$\frac{y}{x} = \frac{3+7}{10} \quad ; \quad \frac{y}{x} = \frac{3-7}{10}$$

$$= \frac{10}{10} \quad ; \quad = -\frac{4}{10}$$

$$\frac{y}{x} = 1 \quad ; \quad \frac{y}{x} = -\frac{2}{5}$$

$$y = 1(x) \quad ; \quad 5y = -2x$$

$$\Rightarrow x - y = 0 \quad \text{(ii)} \quad ; \quad 2x + 5y = 0 \quad \text{(iii)}$$

So, (ii) and (iii) are the required lines.

4. Write short answers to any NINE (9) questions: (18)

(i) Graph the inequality $5x - 4y \leq 20$.

Ans

$$5x - 4y \leq 20 \qquad \text{(i)}$$

(i) can be written as

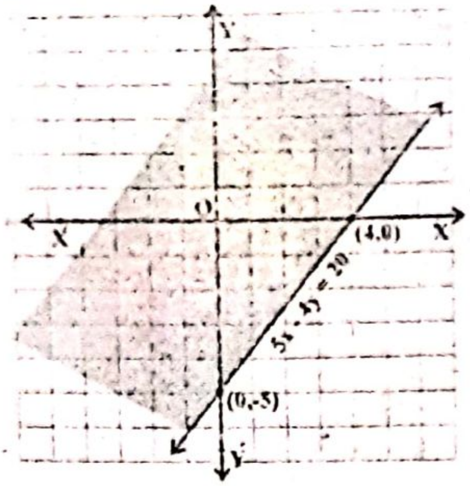
$$5x - 4y < 20 \qquad \text{(ii)}$$

and

$$5x - 4y = 20 \qquad \text{(iii)}$$

(iii) is the corresponding equation of (i) and it is graphed by joining the points (4, 0) and (0, –5) because the points (4, 0), $\left(2, \frac{-5}{2}\right)$, (0, –5), etc. are on the line (iii).

Putting x = 0, y = 0 in the expression 5x – 4y, we get 5(0) – 4(0) = 0 < 20 which shows that its graph is on the origin side of the line (iii), that is, the graph is the open half plane above the boundary line (iii). Thus the graph of the solution set of the linear inequality (i) is the closed half plane.



(ii) Find the equation of the circle with ends of diameter at (–3, 2) and (5, –6).

Ans The centre of a circle is also the mid-point of the end points of diameter, so centre O is at,

$$\left(\frac{-3+5}{2}, \frac{2+(-6)}{2}\right) = \left(\frac{2}{2}, \frac{-4}{2}\right) = (1, -2)$$

$$\text{Radius} = \sqrt{(5-1)^2 + \{-6-(-2)\}^2}$$

$$= \sqrt{(4)^2 + (-6+2)^2} = \sqrt{(4)^2 + (-4)^2}$$

$$= \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$$

Thus an equation of the required circle is

$$(x-1)^2 + (y+2)^2 = (4\sqrt{2})^2$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 32$$

$$x^2 + y^2 - 2x + 4y - 27 = 0$$

(iii) Find the centre of the circle $4x^2 + 4y^2 - 8x + 12y - 25 = 0$.

Ans $4x^2 + 4y^2 - 8x + 12y - 25 = 0$ (i)

Dividing both sides of (i) by 4, we have

$$x^2 + y^2 - 2x + 3y - \frac{25}{4} = 0$$

or $$x^2 + y^2 + 2(-1)x + 2\left(\frac{3}{2}\right)y - \frac{25}{4} = 0 \quad \text{(ii)}$$

Comparing (ii) with $x^2 + y^2 + 2gx + 2fy + c = 0$ gives

$$g = -1 \qquad f = \frac{3}{2} \quad \text{and} \quad c = -\frac{25}{4}$$

Thus the centre of the circle (i) is at $\left(-(-1), -\frac{3}{2}\right) = \left(1, \frac{-3}{2}\right)$

and $r = \sqrt{(-1)^2 + \left(\frac{3}{2}\right)^2 - \left(-\frac{25}{4}\right)}$ $\quad (\because \ r = \sqrt{g^2 + f^2 - c})$

Scanned with CamScanner

$$= \sqrt{1 + \frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{4 + 9 + 25}{4}}$$

$$= \sqrt{\frac{2 \times 19}{2 \times 2}} = \sqrt{\frac{38}{4}} = \sqrt{\frac{19}{2}}$$

(iv) Find the length of the tangent from the point (–5, 10) to the circle $5x^2 + 5y^2 + 14x - 12y - 10 = 0$.

Ans Equation of the given circle in standard form is

$$x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0 \qquad \text{(i)}$$

Square of the length of the tangent from P(–5, 10) to the circle (i) is obtained by substituting –5 for x and 10 for y in the left hand member of (i).

$\therefore$ Required length $= \sqrt{(-5)^2 + (10)^2 - 14 + 24 - 2} = \sqrt{133}$

(v) Find the coordinates of the points of intersection of the line $x + 2y = 6$ with the circle $x^2 + y^2 - 2x - 2y - 39 = 0$.

Ans Given $x^2 + y^2 - 2x - 2y - 39 = 0$ (i)

$x + 2y = 6$ (ii)

From (ii) $2y = 6 - x \Rightarrow y = \frac{6 - x}{2}$ (iii)

Putting the value of y in (i), we have

$$x^2 + \left(\frac{6 - x}{2}\right)^2 - 2x - 2\left(\frac{6 - x}{2}\right) - 39 = 0$$

or $\quad x^2 + \frac{36 - 12x + x^2}{4} - 2x - (6 - x) - 39 = 0$

$\Rightarrow \quad 4x^2 + 36 - 12x + x^2 - 8x - 24 + 4x - 156 = 0$

$\Rightarrow \quad 5x^2 - 16x - 144 = 0$ (iv)

Solving (iv), we have

$$x = \frac{16 \pm \sqrt{256 + 2880}}{10} = \frac{16 \pm \sqrt{3136}}{10} = \frac{16 \pm 56}{10}$$

$\Rightarrow \quad x = \frac{16 + 56}{10} = \frac{72}{10} = \frac{36}{5}$ or $x = \frac{16 - 56}{10} = \frac{-40}{10} = -4$

If $\quad x = -4$, then $y = \frac{6 - (-4)}{2} = \frac{10}{2} = 5$

If $\quad x = \frac{36}{5}$, then $y = \frac{6 - \frac{36}{5}}{2} = \frac{\frac{-6}{5}}{2} = \frac{-6}{5} \times \frac{1}{2} = -\frac{3}{5}$

Thus the points of intersection are $(-4, 5), \left(\frac{36}{5}, \frac{-3}{5}\right)$.

(vi) Find the vertex of the parabola $x^2 = 4(y - 1)$.

Ans Given $x^2 = 4(y-1)$

$$(x-0)^2 = 4(y-1) \quad \text{(i)}$$

By comparing (i) with

$$(x-h)^2 = 4a(y-k) \quad \text{(ii)}$$

We get $4a = 4$

$\Rightarrow$ $a = 1$

So, the vertex of the parabola $x^2 = 4(y - 1)$ is 1.

(vii) Find the foci of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$.

Ans

$$\frac{y^2}{16} - \frac{x^2}{9} = 1 \quad \text{(i)}$$

As the positive term of (i) is $\frac{y^2}{16}$, so the transverse axis of (i) is along the y-axis.

The centre of the hyperbola (i) is (0, 0)

From (i), $a^2 = 16 \Rightarrow a = 4$ and $b^2 = 9 \Rightarrow b = 3$

We know that $c^2 = a^2 + b^2$, that is,

$$c^2 = 16 + 9 = 25 \Rightarrow c = 5$$

Eccentricity $e = \frac{c}{a} = \frac{5}{4}$

As the transverse axis of (i) is along the y-axis, so foci of (i) are $(0, \pm 5)$.

(viii) Find a unit vector in the direction of $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$.

Ans

$$\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$$

$$|\underline{v}| = \left|-\frac{\sqrt{3}}{2}\underline{i} + \left(\frac{-1}{2}\underline{j}\right)\right| = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2}$$

$$\Rightarrow \quad |\underline{v}| = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$$

Unit vector in the direction of $\underline{v} = \frac{\underline{v}}{|\underline{v}|}$

$$= \frac{\frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = \frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}.$$

(ix) Find a vector whose magnitude is 4 and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$.

Ans Let $\underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$

$\Rightarrow \quad |\underline{v}| = |2\underline{i} - 3\underline{j} + 6\underline{k}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{4 + 9 + 36}$

$= \sqrt{49} = 7$

If $\hat{\underline{v}}$ is the unit vector in the direction of $\underline{v}$.

Then $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}$

Thus the required vector is $4\left(\frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}\right)$

i.e., $\frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$

(x) If $\underline{v}$ is a vector for which $\underline{v}.\underline{i} = 0$, $\underline{v}.\underline{j} = 0$ and $\underline{v}.\underline{k} = 0$, find $\underline{v}$.

Ans Let $\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$, then

$\Rightarrow \quad \underline{v} . \underline{i} = (x\underline{i} + y\underline{j} + z\underline{k}) . \underline{i}$

$= x\underline{i} . \underline{i} = x$

As $\quad \underline{v} . \underline{i} = 0$, so $x = 0$

$\underline{v} . \underline{j} = (x\underline{i} + y\underline{j} + z\underline{k}) . \underline{j} = y . \underline{j} . \underline{j} = y$

As $\quad \underline{v} . \underline{j} = 0$, so $y = 0$

$\underline{v} . \underline{k} = (x\underline{i} + y\underline{j} + z\underline{k}) . \underline{k} = \underline{z}\ \underline{k}.\underline{k} = z$

As $\quad \underline{v} . \underline{k} = 0$, so $z = 0$

Thus $\quad \underline{v} = x\underline{i} + y\underline{j} + z\underline{k} = 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$

$\Rightarrow \quad \underline{v} = \underline{0}$ (null vector)

(xi) If $\underline{a} + \underline{b} + \underline{c} = 0$, then prove that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$.

Ans Since $\underline{a} + \underline{b} + \underline{c} = 0$

Taking cross product with $\underline{a}$, we have

$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times 0$

$\Rightarrow \quad \underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$ (Using distribution property)

$\Rightarrow \quad \underline{0} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$

$\Rightarrow \quad \underline{a} \times \underline{b} = -(\underline{a} \times \underline{c})$

$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \qquad$ (i)

Now taking cross product with $\underline{b}$, we have

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times 0$$

$\Rightarrow \quad \underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = 0$ (By distributive property)

$\Rightarrow \quad -(\underline{a} \times \underline{b}) + 0 + \underline{b} \times \underline{c} = 0$

$\Rightarrow \quad \underline{b} \times \underline{c} = \underline{a} \times \underline{b}$ (ii)

From (i) and (ii), we conclude that

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

(xii) Find the volume of parallelepiped for which the vectors $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$, $\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$ and $\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$ are three edges.

Ans Volume of parallelepiped = $\underline{u} \cdot \underline{v} \times \underline{w} = [\underline{w}\ \underline{u}\ \underline{v}]$

$$\Rightarrow \quad \underline{u} \cdot \underline{v} \times \underline{w} = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & +1 \end{vmatrix}$$

$$= 1(-1-6) + 4(1+4) - 1(-3+2)$$

$$= -7 + 20 + 1$$

Volume of parallelepiped = 14

(xiii) Give a force $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point A(1, –2, 1). Find the moment of $\underline{F}$ about the point B(2, 0, –2).

Ans Here $\quad \underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$

$$\underline{r} = \overrightarrow{BA} = (1-2)\,\underline{i} + (-2-0)\,\underline{j} + (1+2)\,\underline{k}$$

$\Rightarrow \quad \underline{r} = -\underline{i} - 2\underline{j} + 3\underline{k}$

Moment of force about B = $\underline{r} \times \underline{F}$

$$\text{Now } \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$$

$$= 3\underline{i} + 3\underline{j} + 3\underline{k}$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) Discuss the continuity of f(x) at x = c **(5)**

$$f(x) = \begin{cases} 3x-1 & \text{if} \quad x < 1 \\ 4 & \text{if} \quad x = 1 \\ 2x & \text{if} \quad x > 1 \end{cases}, \quad c = 1$$

Ans Let $x = 1 + h$

Then $h \to 0^-$ if $x \to 1^-$

and $h \to 0^+$ if $x \to 1^+$

Thus $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x - 1)$

$$= \lim_{h\to 1^-} (3 + 3h - 1)$$

$$= 3 - 1 = 2$$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x)$

$$= \lim_{h\to 0^+} (2 + 2h)$$

$$= 2$$

As $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$, so

$\lim_{x\to 1} f(x) = 2$

But $f(1) = 4$,

which is $\lim_{x\to 1} f(x) \neq f(1)$

Thus f(x) is the discontinuous at x = 1.

(b) Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1}\frac{x}{y}$. **(5)**

Ans Let $\frac{y}{x} = u$ (i)

Then $u = \tan^{-1}\left(\frac{1}{u}\right)$

and $\frac{du}{dx} = \frac{1}{1 + \left(\frac{1}{u}\right)^2}\left(-\frac{1}{u^2}\right)\frac{du}{dx}$

$$= \frac{1}{\frac{u^2 + 1}{u^2}}\left(-\frac{1}{u^2}\right)\frac{du}{dx}$$

$$= \frac{u^2}{u^2 + 1}\left(-\frac{1}{u^2}\right)\frac{du}{dx}$$

$$= \frac{-1}{1 + u^2}\frac{du}{dx}$$

$$\frac{du}{dx} + \frac{1}{1 + u^2}\frac{du}{dx} = 0$$

$$\left(1+\frac{1}{1+u^2}\right)\frac{du}{dx}=0$$

$$\frac{du}{dx}=0$$

But from (i),

$$\frac{du}{dx}=\frac{x\frac{dy}{dx}-y\,.\,(1)}{x^2}$$

$$x^2\frac{du}{dx}=x\frac{dy}{dx}-y$$

$$x^2(0)=x\frac{dy}{dx}-y \qquad \left(\because\ \frac{du}{dx}=0\right)$$

$$0=x\frac{dy}{dx}-y$$

$$y=x\frac{dy}{dx}$$

$$\frac{y}{x}=\frac{dy}{dx}$$

Q.6.(a) Evaluate $\int x\sin^{-1}x\,dx$. **(5)**

Ans

$$=\sin^{-1}x\,.\left(\frac{x^2}{2}\right)-\int\left(\frac{x^2}{x}\right).\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\sin^{-1}x+c$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{1}{2}\sin^{-1}x+\frac{x}{2}\sqrt{1-x^2}\right]-\frac{1}{2}\sin^{-1}x+c$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{4}\left[\sin^{-1}x+x\sqrt{1-x^2}-2\sin^{-1}x\right]+c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{4}\left(-\sin^{-1} x + x\sqrt{1 - x^2}\right) + c$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{x\sqrt{1 - x^2}}{4} + c$$

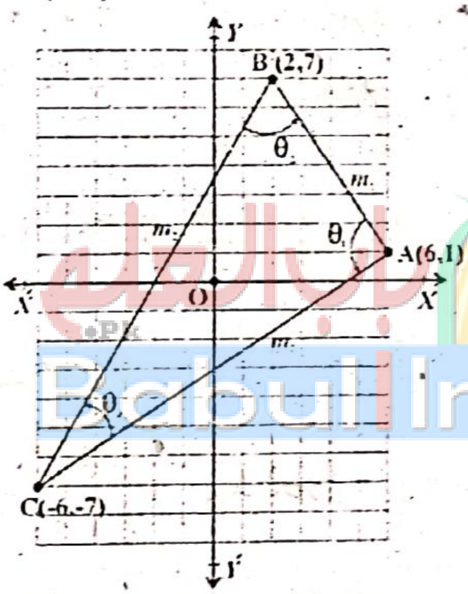
(b) Find the interior angles of the triangle with vertices A(6, 1), B(2, 7), C(–6, –7). **(5)**

Ans Let m_1, m_2 and m_3 be the slopes of the sides AB, BC and CA. Then

$$m_1 = \frac{7 - 1}{2 - 6} = \frac{6}{-4} = -\frac{3}{2}$$

$$m_2 = \frac{-7 - 7}{-6 - 2} = \frac{-14}{-8} = \frac{7}{4}$$

$$m_3 = \frac{-1 - (-7)}{6 - (-6)} = \frac{8}{12} = \frac{2}{3}$$



θ_1, θ_2 and θ_3 are shown in the figure.

$$\tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 \cdot m_1} = \frac{\frac{2}{3} - \left(-\frac{3}{2}\right)}{1 + \left(\frac{2}{3}\right)\left(-\frac{3}{2}\right)} = \frac{\frac{2}{3} + \frac{3}{2}}{1 - \frac{6}{6}} = \frac{\frac{4 + 9}{6}}{1 - 1}$$

undefined

$$\Rightarrow \quad \theta_1 = 90^\circ$$

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 \cdot m_2} = \frac{-\frac{3}{2} - \frac{7}{4}}{1 + \left(-\frac{3}{2}\right)\left(\frac{7}{4}\right)} = \frac{\frac{-6 - 7}{4}}{\frac{8 - 21}{8}}$$

$$= -\frac{13}{4} \times \left(-\frac{8}{13}\right) = 2$$

$$\Rightarrow \quad \theta_2 \approx 63.4^\circ$$

$$\tan\ \theta_3 = \frac{m_2 - m_3}{1 + m_2 \cdot m_3} = \frac{\frac{7}{4} - \frac{2}{3}}{1 + \left(\frac{7}{4}\right)\left(\frac{2}{3}\right)} = \frac{\frac{21-8}{12}}{\frac{12+14}{12}} = \frac{13}{26} = \frac{1}{2}$$

$$\Rightarrow \quad \theta_3 \approx 26.6^\circ$$

Q.7.(a) Evaluate $\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx$. **(5)**

Ans $\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx = \int_0^{\pi/4} \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}\,dx$

$$= \int_0^{\pi/4} \frac{(1-\sin x)}{(1-\sin^2 x)}\,dx$$

$$= \int_0^{\pi/4} \frac{(1-\sin x)}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4} \left[\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right] dx$$

$$= \int_0^{\pi/4} \sec^2 x\,dx - \int_0^{\pi/4} \sec x \tan x\,dx$$

$$= \left|\tan x\right|_0^{\pi/4} - \left|\sec x\right|_0^{\pi/4}$$

$$= \tan\frac{\pi}{4} - \tan 0 - \left(\sec\frac{\pi}{4} - \sec(0)\right)$$

$$= 1 - 0 - [\sqrt{2} - 1] = 1 - \sqrt{2} + 1$$

$$= 2 - \sqrt{2}$$

(b) Minimize $z = 2x + y$ subject to constraints **(5)**

$x + y \geq 3$, $7x + 5y \leq 35$; $x \geq 0$, $y \geq 0$

Ans

$x + y \geq 3$ (i)

$7x + 5y \leq 35$ (ii)

As $1.0 + 1.0 = 0 \ngtr 3$, so the graph of the linear inequality (i) is the closed half plane not on the origin side of $x + y = 3$ which is partially shown by shading in figure (1).

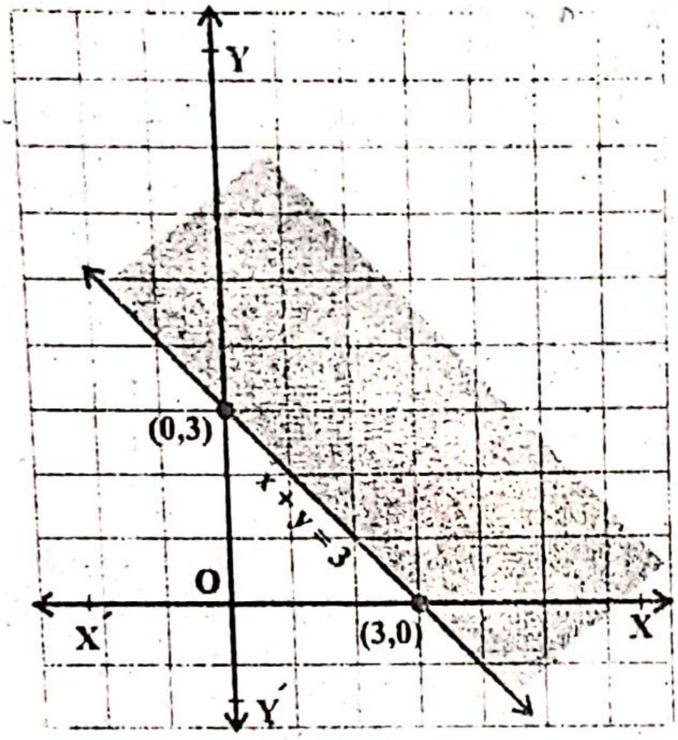


Fig. 1.

The graph of the linear inequality (ii) is the closed half plane on the origin side of $7x + 5y = 35$ which is partially indicated by shading in figure (2).

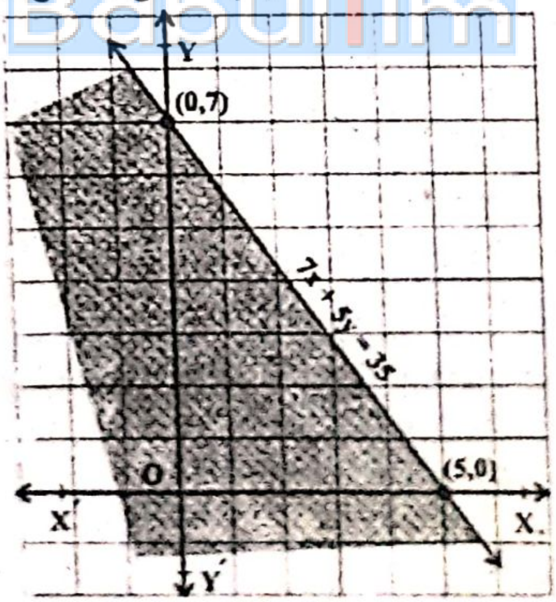


Fig. 2.

The intersection of graphs partially shown in figures (i) and (ii) is the solution region of the inequalities (i) and (ii) which is partially indicated by shading in figure (3).

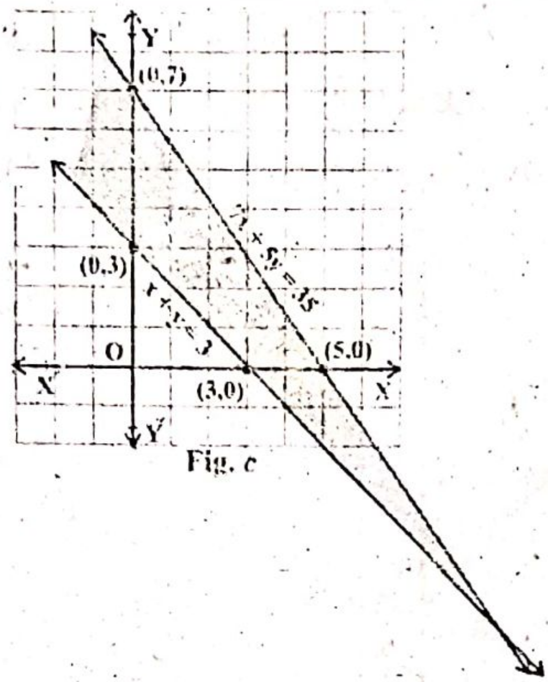


Fig. 3.

Using the non-negative constraints, the feasible region is shown in figure (4). The corner points of the feasible region are (3, 0), (5, 0), (0, 7) and (0, 3).

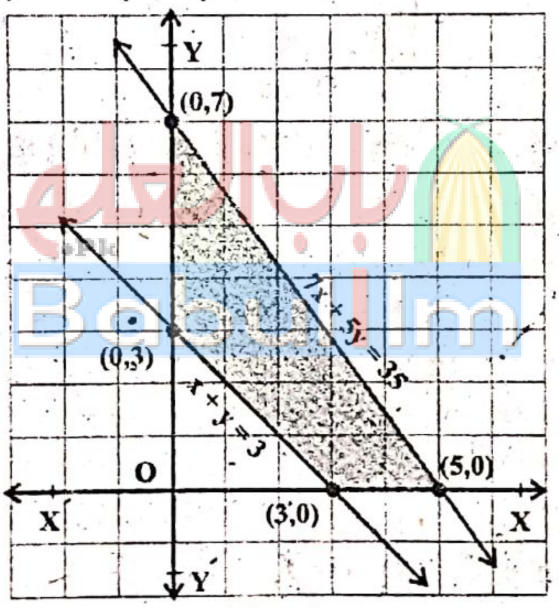


Fig. 4.

Let z be written as $\phi(x, y)$ that is, $z = \phi(x, y) = 2x + y$

$\phi(3, 0) = 2(3) + 0 = 6$

$\phi(5, 0) = 2(5) + 0 = 10$

$\phi(0, 7) = 2(0) + 7 = 7$

$\phi(0, 3) = 2(0) + 3 = 3$

z is minimum at the corner point (0, 3).

Q.8.(a) Prove that in any triangle ABC $b^2 = c^2 + a^2 - 2ca \cos B$. (5)

Ans Let vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ be along the sides BC, CA and AB of triangle ABC, respectively.

Since $\quad \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = 0$

$\Rightarrow \quad \underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \quad \underline{b} = -(\underline{a} + \underline{c})$

Now $\quad \underline{b} \cdot \underline{b} = -(\underline{a} + \underline{c}) \cdot [-(\underline{a} + \underline{c})]$

$b^2 = (\underline{a} + \underline{c}) \cdot (\underline{a} + \underline{c})$

$\underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{c}$

$a^2 + 2\underline{a} \cdot \underline{c} + c^2$

$b^2 = a^2 + 2\underline{a} \cdot \underline{c} + c^2$

$2(\underline{a} \cdot \underline{c})$

$\Rightarrow \quad b^2 = a^2 + c^2 + 2\underline{c} \cdot \underline{a}$

$\Rightarrow \quad b^2 = a^2 + c^2 + 2ca \cos(\pi - B)$

$\Rightarrow \quad b^2 = a^2 + c^2 + 2ac(-\cos B)$

$\Rightarrow \quad b^2 = a^2 + c^2 - 2ca \cos B$

(b) Find the length of the chord cut off from the line $2x + 3y = 13$ by the circle $x^2 + y^2 = 26$. (5)

Ans $\quad x^2 + y^2 = 26 \quad$ (i)

and $\quad 2x + 3y = 13 \quad$ (ii)

From (ii) $3y = 13 - 2x \Rightarrow y = \dfrac{13 - 2x}{3} \quad$ (iii)

Putting the value of y in (i), we get

$$x^2 + \left(\frac{13 - 2x}{3}\right)^2 = 26 \quad \text{or} \quad x^2 + \frac{169 - 52x + 4x^2}{9} = 26$$

$\Rightarrow \quad 9x^2 + 4x^2 - 52x + 169 - 234 = 0$

$13x^2 - 52x - 65 = 0 \Rightarrow x^2 - 4x - 5 = 0 \quad$ (iv)

Solving (iv), we have $x = \dfrac{4 \pm \sqrt{16 + 20}}{2} = \dfrac{4 \pm \sqrt{36}}{2} = \dfrac{4 \pm 6}{2}$

$\Rightarrow \quad x = \dfrac{4 + 6}{2} = 5 \quad \text{or} \quad x = \dfrac{4 - 6}{2} = \dfrac{-2}{2} = -1$

If $\quad x = 5$ then $y = \dfrac{13 - 2(5)}{3} = \dfrac{13 - 10}{3} = \dfrac{3}{3} = 1$

and if $x = 1$, then $y = \dfrac{13 - 2(-1)}{3} = \dfrac{13 + 2}{3} = \dfrac{15}{3} = 5$

Thus the points of intersection are (5, 1), (–1, 5)
Length of the chord between the points of intersection

$$= \sqrt{\{5-(-1)^2\} + (1-5)^2} = \sqrt{(6)^2 + (-4)^2}$$
$$= \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$$

Q.9.(a) If $y = (\cos^{-1} x)^2$, then prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$. (5)

Ans Given: $y = (\cos^{-1} x)^2$

$$\frac{dy}{dx} = y_1 = \frac{d}{dx}(\cos^{-1} x)^2$$
$$= 2(\cos^{-1} x)\frac{d}{dx}(\cos^{-1} x)$$
$$= 2\cos^{-1} x\left(\frac{-1}{\sqrt{1-x^2}}\right)$$
$$y_1 = \frac{-2\cos^{-1} x}{\sqrt{1-x^2}}$$
$$\sqrt{1-x^2}\, y_1 = -2\cos^{-1} x$$

By taking derivative again,

$$\frac{-2x}{2\sqrt{1-x^2}}y_1 + \left(\sqrt{1-x^2}\right)y_2 = -2\left(\frac{-1}{\sqrt{1-x^2}}\right)$$
$$\frac{-x}{\sqrt{1-x^2}}y_1 + \sqrt{1-x^2}\, y_2 = \frac{2}{\sqrt{1-x^2}}$$
$$\sqrt{1-x^2} - \frac{xy_1}{\sqrt{1-x^2}} + \sqrt{1-x^2}\sqrt{1-x^2}\, y_2 = \frac{2}{1-x^2}\left(\sqrt{1-x^2}\right)$$
$$-xy_1 + (1-x^2)y_2 = 2$$
$$(1-x^2)y_2 - xy_1 - 2 = 0 \quad \text{Proved.}$$

(b) Find the points of intersection of the given conic $\frac{x^2}{18} + \frac{y^2}{8} = 1$ and $\frac{x^2}{3} - \frac{y^2}{3} = 1$. (5)

Ans

$$\frac{x^2}{18} - \frac{y^2}{8} = 1 \qquad \text{(i)}$$
$$\frac{x^2}{3} - \frac{y^2}{3} = 1 \qquad \text{(ii)}$$

From (ii), $\frac{y^2}{3} = \frac{x^2}{3} - 1 = \frac{x^2-3}{3} \Rightarrow y^2 = x^2 - 3$ (iii)

Putting $y^2 = x^2 - 3$ in (i), we get

$$\frac{x^2}{18}+\frac{x^2-3}{8}=1 \Rightarrow \frac{4x^2+9x^2-27}{72}=1$$

or $\quad 13x^2-27=72 \Rightarrow 13x^2=99$

or $\quad x^2=\frac{99}{13} \qquad \Rightarrow x=\pm\sqrt{\frac{99}{13}}$

Putting $x=\sqrt{\frac{99}{13}}$ in (iii), we have

$$y^2=\frac{99}{13}-3=\frac{99-39}{13}=\frac{60}{13} \Rightarrow y=\pm\sqrt{\frac{60}{13}}$$

Putting $x=-\sqrt{\frac{99}{13}}$ in (iii), we get

$$y^2=\left(-\sqrt{\frac{99}{13}}\right)^2-3=\frac{99}{13}-3=\frac{99-39}{13}=\frac{60}{13}$$

$\Rightarrow \quad y=\pm\sqrt{\frac{60}{13}}$

Thus the point of intersections are $\left(\frac{\sqrt{99}}{3}, \frac{\sqrt{60}}{13}\right)$

$\left(\sqrt{\frac{99}{3}}, -\sqrt{\frac{60}{13}}\right), \left(-\sqrt{\frac{99}{3}}, \sqrt{\frac{60}{13}}\right)$ and

$\left(-\sqrt{\frac{99}{3}}, -\sqrt{\frac{60}{13}}\right)$